On Forces in Magnetized Matter

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Abstract—In magneto-elastic interactions, the Virtual Power Principle is powerful enough to solve all force problems, but only if the coupled constitutive laws are known integrally. We discuss the problem of forces in magnetized matter, insisting on the fact that a full knowledge of the magnetic field may not be enough to determine the force density when the local $B-H$ law depends on the local strain of the material.

Index Terms—Electromagnetic forces, Maxwell tensor, magnetostriction.

I. INTRODUCTION

Forces in magnetized matter are not yet properly understood. Not, at least, from the point of view of engineers involved in calculations. We should like to have formulas that, once the field has been computed by finite elements or other numerical techniques, could be coded in order to compute the force density from the fields data. There is no dearth of such formulas, but their amazing diversity is disturbing: Are all these proposals equivalent, in spite of formal differences, or are they substantially distinct, being based on different theories?

Part of the difficulty—the mildest part—is notational: It could be the case that two formulas are secretly the same, being reducible one to the other by a few lines of vector algebra, but checking that may be difficult, or it may be so hard as to disagree? Which does happen, as we well know: The field $M$ inside a hard magnet can be ascribed to magnetic charges or to Amperian currents, and hence, to different microscopic models: For instance, magnetized matter is constricted as a distribution of magnetic dipoles, and since the mechanical effect of the field on such dipoles is known, one may think that summing up these elementary forces solves the problem. But what if different microscopic models then appear to disagree? Which does happen, as we well know: The field $M$ inside a hard magnet can be ascribed to magnetic charges or to Amperian currents, with identical results as regards the total force and torque on the magnet, but with spectacularly different predictions about the force field, and hence, about the eventual deformation of the magnet. So one should not rely on imagined microscopic mechanisms, to develop the theory, only on measurable macroscopic properties: The virtual power principle (VPP) makes this possible, and points to the determination of the energy density as a function of both the electromagnetic field (EM field) and the material deformation as the central problem in the question of forces.

At this stage, one realizes that knowing the EM field may not be enough to compute the force field: a complete description of the coupled constitutive laws may be required. As a corollary, the Maxwell tensor does not know all about forces, which challenges its (alleged by some) status as cornerstone of the theory: Indeed, there are cases when the force density differs from the divergence of the Maxwell tensor. This is the hallmark of magnetostriction: The state of affairs when local magnetic properties such as permeability or magnetization depend on the local deformation of the body.

The purport of the present paper is to unfold this theory from first principles, using the lightest possible mathematical apparatus. In particular, the “material form” [1] of the equations and the differential-geometric formalism that naturally goes with it [2] are avoided, in favor of a Eulerian treatment with familiar vector entities only. We first reestablish a standard result: Force is equal to $J \times B$ minus the derivative of magnetic energy, expressed as a function of $B$ and the displacement field, with respect to the latter. Then we treat a few simple cases, enough to show how easily can magnetostriction be overlooked, and how the formalism allows to take it into account when needed.

II. GENERAL EXPRESSION OF THE FORCE FIELD

We work in 3D Euclidean space, with dot product $X \cdot Y$. A material particle lying at point $x$ at time $t = 0$ will be found at point $x + u(t,x)$ at time $t$, the displacement $u$ being a smooth vector-valued function of $x$ and $t$, null for $t = 0$. (We don’t want initially distinct particles to collide, so the correspondence $x \rightarrow x + u(t,x)$ should be 1–1: This is so for $t > 0$ small enough.) The velocity field $v$ (set in roman to avoid possible confusion with reluctivity $\nu$) is the time derivative $\partial_t u$. Only its value at time $0$ will matter when invoking the VPP. One assumes $u(t,x)$ uniformly bounded and null outside some bounded region of space that contains all the matter and currents participating in the interaction. (For convenience, points $x$ of this region that lie in the air are also assigned a displacement, whose value will play no role, provided $u$ stays smooth.) A source-current density $J'(t,x)$, maintained by some exterior agency, is given, also null far away. Time-dependent fields $E$, $H$, $B$ describe the electromagnetic situation. (The displacement current $D$ is ignored, as well as electric charge, which entails the neglect of Coulomb forces.)

We adopt the time primitive (up to sign) of the electric field, $A(t) = A_0 - \int_0^t E(s) \, ds$, as field descriptor. Thus, $E = -\partial_t A$ and $B = \text{rot} \, A$. Conductivity and reluctivity (more convenient here than its inverse, the permeability $\mu$) will depend on the displacement $u$, so we denote them by $\sigma_u$ and $\nu_u$, without being more specific for the time being. A typical form of the
evolution equation is, with initial condition $A(0) = A_0$,
\[
\sigma_v(\partial_v A - \nu \times B) + \nu (\nu_v \cdot \partial_v A) = J',
\]
but to gain some generality we shall introduce the magnetic energy $\Psi(u, A)$, equal to $\frac{1}{2} \int \nu_v |\partial_v A|^2$ in the case of (1). (The integration, here and in what follows, is over all space.) This way the partial Fréchet derivative $\partial_v \Psi$ is the vector field $\partial_v H$, where $H = \nu_v \cdot \partial_v A = \nu_v B$, so the equation becomes
\[
\sigma_v(\partial_v A - \nu \times B) + \partial_v \Psi(u, A) = J',
\]
which covers the nonlinear case of ferromagnetic (but non-hysteretic) materials.

Since $\Psi$ appears here as just a device to formulate the magnetic law, we should justify calling it magnetic energy. For this, remark that the rate of change of $\Psi(u, A)$ is, by the chain rule,
\[
\frac{d}{dt}(\Psi(u(t), A(t))) = \int \partial_v \Psi \cdot v + \int \partial_v A \cdot \partial_v A.
\]
Next, taking $v = 0$ in (2), dot-multiply both sides of it by $\partial_v A$ and integrate over space. This results in
\[
\int \sigma_v |\partial_v A|^2 + \frac{d}{dt}(\Psi(u, A(t))) = -\int J'(t) \cdot E(t).
\]

The right-hand side of (4) is the power brought into the system by the source current, and $\int \sigma_v |\partial_v A|^2 = \int \sigma_v |E|^2$ is the Joule loss. So the second term on the left represents the fraction of power that must be stored in the magnetic field, which is the needed justification: With the convention that $\Psi(u, 0) = 0$ whatever $u$, the quantity $\Psi(u, A)$ appears as the magnetic potential of the field $B = \nu \cdot \partial_v A$ in configuration $u$.

Now, return to the case where $u$ can evolve in time. Denoting by $J = \sigma_v(-\partial_v A + \nu \times B)$ the induced current in conductors, Joule losses are
\[
\int \sigma_v |\partial_v A - \nu \times B|^2 = \int \sigma_v |\partial_v A - \nu \times B| \cdot \partial_v A - \int v \cdot (J \times B).
\]
Repeat the above process—dot-multiply both sides of (2) by $\partial_v A$ and integrate in space. Combining (3) and (5), one gets
\[
\int \sigma_v |\partial_v A - \nu \times B|^2 + \frac{d}{dt}(\Psi(u, A)) + \cdots + \int v \cdot (J \times B - \partial_v \Psi(u, A)) = -\int J' \cdot E.
\]

Considering $v$ here as the velocity in a virtual motion, the third term on the left appears as the corresponding virtual power. The force field, therefore, is $J \times B - \partial_v \Psi(u, A)$.

The full paper will show how the vector field $\partial_v \Psi$ is computed in practice, in a series of important cases: Nonhomogeneous B–H law, linear or nonlinear, isotropic or anisotropic, permanent magnets. This summary must restrict to the first topic. Yet there is room for a remark of general validity: We are interested in the force field in the reference configuration, the one in which $u = 0$, so we only want $\partial_v \Psi(u, A)$ at $u = 0$ (and $A = A_0$, the initial condition). In such a case the following trick, based on the concept of “directional” derivative, is available: For each vector field $v$ (conceived here as a virtual velocity field), set $u = rv$, and find the limit of $[\Psi(rv, A_0) - \Psi(0, A_0)]/t$ when $t$ tends to 0. This limit, which is also the derivative in $t$ of $\Psi(rv, A_0)$ at $t = 0$, has the form $\int v \cdot \partial_v \Psi(0, A_0)$, from which what we shall call the “extra force field”, $f = -\partial_v \Psi(0, A_0)$, can be read off.