Dual Formulations for Accurate Thin Shell Models in a Finite Element Subproblem Method

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Abstract—Subproblem dual finite element magnetostatic and magnetodynamic formulations are developed and compared for correcting the inaccuracies near edges and corners of thin shell models, that replace thin volume regions by surfaces. The surface-to-volume correction problem is defined as one of the multiple subproblems applied to a complete problem, considering successive additions of inductors and magnetic or conducting regions, some of these being thin regions. Each SP requires a proper adapted mesh of its regions, which facilitates meshing and increases computational efficiency.

I. INTRODUCTION

As proposed in [1], [2], thin shell (TS) finite element (FE) models are used to avoid meshing thin regions (TRs) and lighten the mesh of their surrounds. For that, the volume TRs are replaced by surfaces with interface conditions (ICs) linked to 1-D analytical distributions along their thickness that however neglect end and curvature effects. This leads to inaccuracies of field distributions and associated losses near edges and corners, that increase with the thickness. To overcome these disadvantages, a subproblem method (SPM) based on magnetic flux density formulations, proposing a surface-to-volume local correction, has been proposed in [3].

The SPM for TS correction is herein extended to a dual approach for both magnetic field and magnetic flux density formulations, with generalized mesh projections of solutions between the SPs. Also, the SPM naturally allows parameterized analyses of the TR characteristics: permeability, conductivity and thickness. In the proposed SP strategy, a reduced problem (SP u) with only inductors is first solved on a simplified mesh without thin and volume regions. Its solution gives surface sources (SSs) as ICs for added TS regions (SP p), and volume sources (VSs) for possible added volume regions (SP k). The TS solution is then corrected by a volume correction via SSs and VSs that suppress the TS representation and add the volume model.

II. THIN SHELL CORRECTION IN THE SUBPROBLEM METHOD

A. Canonical magnetodynamic or static problem

A canonical magnetodynamic or static problem i, to be solved at step i of the SPM, is defined in a domain Ωi, with boundary ∂Ωi = Γi = Γh,i ∪ Γk,i. Material relations and boundary conditions (BCs) are [3]

\[ \begin{align*}
    h_i &= \mu_i^{-1}b_i + h_{i,k}, \ j_i = \sigma_i e_i + j_{i,k} \\
    n \times h_i|_{\Gamma_k,i} &= j_{f,i}, \ n \times e_i|_{\Gamma_k,i} = k_{f,i}
\end{align*} \]  

(1a-b)

where \( h_i \) is the magnetic field, \( b_i \) is the magnetic flux density, \( e_i \) is the electric field, \( j_i \) is the electric current density, \( \mu_i \) is the magnetic permeability, \( \sigma_i \) is the electric conductivity and \( n \) is the unit normal exterior to \( \Omega \). The notation \( |\cdot|_{\Gamma_k,i} = |\cdot|_{\Gamma_k,i} - |\cdot|_{\Gamma_h,i} \) refers to the discontinuity of a quantity through an interface \( \gamma_i \) (with sides \( \gamma_i^+ \) and \( \gamma_i^- \)) in \( \Omega_i \), defining ICs. The fields \( h_{i,k} \) and \( j_{i,k} \) in (1a) and (1b) are VSs that can be used for expressing changes of a material property in a volume region, from \( \mu_p \) and \( \sigma_p \) for SP p to \( \mu_k \) and \( \sigma_k \) for SP k [3], i.e.

\[ h_{i,k} = (\mu_k^{-1} - \mu_p^{-1})b_p, \ j_{i,k} = (\sigma_k - \sigma_p)e_p \]  

(3)

with \( \mu_p = \mu_0, \mu_k = \mu_{\text{volume}}, \sigma_p = 0 \) and \( \sigma_k = \sigma_{\text{volume}} \). The fields \( j_{f,i} \) and \( k_{f,i} \) in (2a) and (2b) are SSs. They define possible SSs that account for particular phenomena occurring in the idealized TR between \( \gamma_i^+ \) and \( \gamma_i^- \) [3]. This is the case when some field traces in SP p are forced to be discontinuous (e.g. in TS model), whereas their continuity must be recovered via an SP k, which is done via a SS in SP k fixing the opposite of the trace discontinuity solution of SP p.

B. SSs via ICs for subproblems

The solution of an SP u is first known for a particular configuration, e.g. for an inductor alone (Fig. 1, a), or more generally resulting from the superposition of several SP solutions. The next SP p consists in adding a TS to this configuration (Fig. 1, b). From SP u to SP p, the solution u gives SSs for the added TS γp, through TS ICs [2].

Figure 1: Interface condition between SP u and SP p.

b-formulation uses a magnetic vector potential \( a_i \) (such that \( \text{curl} a_i = b_i \)), split as \( a = a_{e,i} + a_{d,i} \) [2]. The h-formulation uses a similar splitting for the magnetic field, \( h_i = h_{e,i} + h_{d,i} \). The fields \( a_{e,i}, h_{e,i} \) and \( a_{d,i}, h_{d,i} \) are continuous and discontinuous respectively through the TS. The trace discontinuities in SP p
\[ [n \times h_p]_y = [n \times e_p]_y, \text{ with } n_i = -n \text{ can be expressed as } \]

\[ [n \times h_1]_y = [n \times (h_1 + h_p)]_y - [n \times h_1]_y = [n \times h]_y \] 

\[ [n \times e_p]_y = [n \times (e_u + e_p)]_y - [n \times e_u]_y = [n \times e]_y \] 

(4) 

(5) 

because there are no discontinuities in SP \( \gamma_p \) (before adding \( \gamma_p \)). 

In addition, one has TS-ICs in both formulations [2] 

\[ [\nabla \times (h_u + h_p)]_y = \mu_0 \beta_p \partial_t (2a_{c,p} + a_{d,p}) \] 

(6) 

\[ [\nabla \times (e_u + e_p)]_y = \mu_0 \beta_p \partial_t (2h_{c,p} + h_{d,p}) \] 

(7) 

with \( \beta_p \) given in [2]. The resulting FE formulations are then written for SPs \( u, p \) and \( k \), which will be developed in the full paper.

\[ \begin{align*}
  b_u, \text{ SP } u \\
  j_p, \text{ SP } p \\
  j_k, \text{ SP } k
\end{align*} \]

**III. Application Example**

A 3-D test problem is based on TEAM problem 21 (model B, coil and plate, Fig. 2). An SP scheme considering three procedures is developed. A first SP \( \text{FE} u \) with the stranded inductors alone is solved on a simplified mesh without any TR (Fig. 2, left). Then an SP \( \text{FE} p \) is solved with the added TR via a TS FE model (Fig. 2, middle). At last, an SP \( k \) replaces the TS FE with the actual volume FEIs (Fig. 2, right). The TS error on \( j_p \) locally reaches 83% (Fig. 2, middle), with \( f = 50 \text{ Hz}, \mu = 200 \text{ and } \sigma = 6.484 \text{ MS/m (skin depth } \delta = 6.15 \text{ mm) . The inaccuracies on the Joule power loss densities of TS SP } p \text{ are pointed out by the importance of the correction SP } k \text{ (Fig. 3, top) . Significant errors on TS SP } p \text{ along the horizontal half inner width (y-direction) reach 85% near the plate ends (Fig. 3, top), with } \delta = 2.1 \text{ mm and thickness } d = 7.5 \text{ mm . For } d = 1.5 \text{ mm, they are reduced to below 10% . In particular, accurate local corrections with volume correction SP } k \text{ are checked to be close to the solution of the complete problem, with errors lower than 0.1% (Fig. 3, bottom) .} 

Table I shows the Joule losses in the plate with approximate BCs for SP \( k \text{. The exterior boundary of SP } k \text{ is first chosen at a distance } D_{\text{bound}} = 200d \text{ from the TR, with thickness } d = 10 \text{ mm . The inaccuracies on Joule losses for TS SP } p \text{ reach 76.4%, or 1.2% for volume correction SP } k \text{, with } f = 50 \text{ Hz, } \mu = 100 \text{ and } \sigma = 6.484 \text{ MS/m in both cases . The proposed SP strategy allows to locally focus on the mesh of volume correction SP } k \text{ and its neighborhood . It is shown that even if } D_{\text{bound}} \text{ is reduced to } 2d \text{, the error on SP } k \text{ is 1.53%, which is still very accurate . For } d = 2 \text{ mm, the errors on Joule losses for SP } p \text{ are reduced to 6.09%, or 0.05% for SP } k .

Table I: Joule losses in the plate with approximate BCs (\( f = 50 \text{ Hz, } \mu = 100 \text{, } \sigma = 6.484 \text{ MS/m) , with } b-\text{formulation})

<table>
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<th>( D_{\text{bound}} ) (thickness of the plate)</th>
<th>( d = 10 \text{ mm)</th>
<th>Errors %</th>
<th>Between } P_{\text{th}} \text{ and } P_{\text{ref}}</th>
<th>\text{ Bet } P_{\text{vol}} \text{ and } P_{\text{ref}}</th>
</tr>
</thead>
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<td>Volume</td>
<td>Reference</td>
<td>( P_{\text{th}} ) and ( P_{\text{ref}} )</td>
<td>( P_{\text{vol}} ) and ( P_{\text{ref}} )</td>
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<td>200d</td>
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**References**

